COMMUTATIVE ALGEBRA MMATH SECOND YEAR - MIDSEMESTER EXAMINATION

Attempt all questions - Total Marks 40 - 10 a.m. to 1 p.m., 17th September 2012

All rings considered here are commutative rings with multiplicative identity.

(1) Let I and J be two ideals of a ring A. Prove that, every prime ideal which contains I also contains J, if and only if, for all $x \in J$ there exists a positive integer n (depending on x) such that $x^n \in I$. If J is a finitely generated ideal, then the above two conditions are furthur equivalent to: there exists a positive integer m such that $J^m \subset I$. (6 marks)

(2) Let $(A, \mathcal{M})$ be a local ring and let M, N be two finitely generated A-modules. Prove that $M \otimes_A N = 0$ implies that either $M = 0$ or $N = 0$. (6 marks)

(3) Let A be a ring and $\mathcal{P}$ be a prime ideal of A. Prove that $A_{\mathcal{P}}/\mathcal{P}A_{\mathcal{P}} \cong Q(A/\mathcal{P})$ (where $Q(A/\mathcal{P})$ denotes the quotient field of the domain $A/\mathcal{P}$). Give all details of your argument. (6 marks)

(4) Let $\phi : A \to B$ be a surjective ring homomorphism and let M, N be two B-modules. Show that $M \otimes_A N \cong M \otimes_B N$ (here M, N are treated as A-modules in the first tensor product, and the isomorphism is as A-modules). Give an example to show that this is not true if ϕ is not surjective. (7 marks)

(5) Let M, N be two A-modules, and let $I \subset A$ be an ideal. Prove that

$$(M \otimes_A N)/I(M \otimes_A N) \cong M/IM \otimes_{A/I} N/IN$$

(7 marks)

(6) Let I be a directed set. Let $(M_i)_{i \in I}$ and $\mu_{ij} : M_i \to M_j$ (for every $i \leq j$) be a direct system of A-modules indexed by I, let M be the direct limit of this direct system (with homomorphisms $\mu_i : M_i \to M$). Show that every element of M can be written as $\mu_i(x_i)$ for $x_i \in M_i$ for some $i \in I$. Show that if $\mu_i(x_i) = 0$ for some $x_i \in M_i$, then $\mu_{ij}(x_i) = 0$ in M_j for some $j \in I$ such that $i \leq j$. (8 marks)